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A NOTE ON BISIMULATIONS OF FINITE KRIPKE MODELS

Abstract

In our paper we consider the notion of bounded bisimulation of Kripke models for intuitionistic first-order theories. As it is already known, in this case, the existence of bisimulation between given two Kripke models implies their logical equivalence. We present a new result which states that, under some additional conditions, for every two first-order Kripke models that are equivalent, there is a bisimulation between them.

Introduction

Given two classical structures, one of the most important questions is whether they validate the same formulae. Since the notion of logical equivalence involves language, our aim is to find a suitable condition that is defined directly in terms of structural properties. When we discard the notion of structure isomorphism as a too strong and too restrictive one, we have to look for another condition for logical equivalence.

In classical model theory the problem was to find a structural description for the notion of elementary equivalence. It was first stated by Albert Tarski. The solution was found by Fraïssé and then by Ehrenfeucht, and can be formulated in terms of the well-known notion of Ehrenfeucht–Fraïssé game. One of the versions of Ehrenfeucht-Fraïssé Theorem states that two classical first-order structures $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent with respect to all sentences with quantifier complexity not greater than n, $\mathcal{A} \equiv_n \mathcal{B}$, whenever there exists a winning strategy for Duplicator in Ehrenfeucht–Fraïssé game of length n on structures $\mathcal{A}$ and $\mathcal{B}$. It turns out that the inverse implication also holds, but it requires some additional assumptions.

In this paper we consider the case of Kripke semantics for intuitionistic first-order theories. In this case the problem is to find a structural description for the notion of logical equivalence of two Kripke models. To this end, as a counterpart of the Ehrenfeucht–Fraïssé game of length n on structures $\mathcal{A}$ and $\mathcal{B}$ we consider the notion of bounded bisimulation between nodes α and β of Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively. Namely, if we confine our considerations to Kripke models built up from a single node, the notion of bisimulation coincides with the Ehrenfeucht–Fraïssé game, and theorem which states that bisimulation implies logical equivalence is an analogue of Ehrenfeucht-Fraïssé Theorem mentioned before.

In the case of Kripke models for intuitionistic first-order logic, it is already known that the existence of bisimulation between nodes of two Kripke models implies their logical equivalence (see [2]). The subject of our research is, however, the reverse implication.

The paper is organised as follows. Section 2 provides an overview of basic definitions needed in further considerations. It contains notions such as first-order formula, a characteristic of a formula, first-order Kripke model as well as logical equivalence, and bounded bisimulation.

In Section 3 we quote the well-known result concerning bounded bisimulation and logical equivalence. Then, we confine ourselves to the case of finite signature of the language L with no function symbols. Moreover, we consider *finitely saturated* Kripke models. Under those assumptions we obtain the main theorem which states that logical equivalence of nodes of two *strongly finite* Kripke models implies bisimulation between them.

Preliminaries

The aim of this section is to present the notion of bisimulation in the case of first-order intuitionistic logic. For a comprehensive overview of classical model theory topics see [1]. Definitions appearing in this section can be found in [3] and [2].

Let us consider the classical first-order language L with equality. Its (possibly infinite) signature consists of constants, function and relation symbols. First-order formulae are built up from atoms and symbols $\bot$, $\top$ (*falsum* and *verum*) by means of $\wedge$, $\vee$, $\rightarrow$, and quantifiers $\exists$, $\forall$. As a measure of formula's complexity, we define the *characteristic* of a formula $\varphi(\overline{x})$, $char(\varphi)$, as follows

- If φ is an atomic formula, then $char(\varphi) = ({}^{\to}0, {}^{\forall}0, {}^{\exists}0)$.

Suppose that formulas φ_1, φ_2 are given and $char(\varphi_i) = ({}^{\to}p_i, {}^{\forall}q_i, {}^{\exists}r_i)$ for $i = 1, 2$. Let $p = \max(p_1, p_2)$, $q = \max(q_1, q_2)$ and $r = \max(r_1, r_2)$.

- If $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi = \varphi_1 \vee \varphi_2$, then $char(\varphi) = ({}^{\to}p, {}^{\forall}q, {}^{\exists}r)$.
- If $\varphi = \varphi_1 \to \varphi_2$, then $char(\varphi) = ({}^{\to}p + 1, {}^{\forall}q, {}^{\exists}r)$.
- If $\varphi = \forall_x \varphi_1$, then $char(\varphi) = ({}^{\to}p_1, {}^{\forall}q_1 + 1, {}^{\exists}r_1)$.
- If $\varphi = \exists_x \varphi_1$, then $char(\varphi) = ({}^{\to}p_1, {}^{\forall}q_1, {}^{\exists}r_1 + 1)$.

We put $({}^{\to}p, {}^{\forall}q, {}^{\exists}r) \preceq ({}^{\to}p', {}^{\forall}q', {}^{\exists}r')$ whenever $p \leq p'$, $q \leq q'$ and $r \leq r'$.

Consider two classical first-order structures M and N for a given language L. A function $f: M \to N$ will be called *weakly structure preserving* if and only if

(i) for every n, every n-ary function symbol F of L, and every n-tuple $\overline{a} \in M$,
$$f(F^M(\overline{a})) = F^N(f\overline{a}),$$

(ii) for every n, every n-ary predicate symbol P of L, and every n-tuple $\overline{a} \in M$,
$$P^M(\overline{a}) \Longrightarrow P^N(f\overline{a}).$$

Let $\mathbb{K}$ be a partial order viewed as a small category. Its objects called *nodes* will be denoted by $\alpha, \beta, \gamma, \delta$, etc., while morphisms between them by f, g, etc. For simplicity, and to emphasize that $\mathbb{K}$ is a partial order, we will write $\alpha \leq^f \beta$ to denote the morphism $f: \alpha \to \beta$ between nodes α and β. Let $\mathbb{A}$ be the category of classical first-order structures with weakly preserving functions as morphisms. By a Kripke model for a first-order language L we mean a functor $\mathcal{K}: \mathbb{K} \to \mathbb{A}$ such that for each arrow $\alpha \leq^f \beta$ there is assigned a weakly structure preserving function $\mathcal{K}(f): \mathcal{K}(\alpha) \to \mathcal{K}(\beta)$. Category $\mathbb{K}$ is called the *frame* of the model $\mathcal{K}$, whereas objects of $\mathbb{A}$ will be called *worlds* of the model $\mathcal{K}$. For simplicity we will write f for $\mathcal{K}(f)$, and K_α for $\mathcal{K}(\alpha)$. So, informally, a Kripke model $\mathcal{K}$ can be viewed as a family of first-order structures partially ordered by weak homomorphisms between them. For the general definition see [4].

The forcing relation $\Vdash_{\mathcal{K}}$ on $\mathcal{K}$ is defined inductively over the construction of a formula. Consider a node α and a sequence $\overline{a} := a_1, \ldots, a_n$ of elements of the structure K_α, we put

- $\alpha \nVdash_{\mathcal{K}} \bot$ and $\alpha \Vdash_{\mathcal{K}} \top$
- $\alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}) \iff K_\alpha \models \varphi(\overline{a})$ for all atomic formulas $\varphi(\overline{x})$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \wedge \psi)(\overline{a}) \iff \alpha \Vdash_{\mathcal{K}} \varphi(\overline{a})$ and $\alpha \Vdash_{\mathcal{K}} \psi(\overline{a})$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \vee \psi)(\overline{a}) \iff \alpha \Vdash_{\mathcal{K}} \varphi(\overline{a})$ or $\alpha \Vdash_{\mathcal{K}} \psi(\overline{a})$
- $\alpha \Vdash_{\mathcal{K}} (\varphi \to \psi)(\overline{a}) \iff \forall_{\alpha \leq^f \alpha'} (\alpha' \Vdash_{\mathcal{K}} \varphi(f\overline{a}) \Rightarrow \alpha' \Vdash_{\mathcal{K}} \psi(f\overline{a}))$
- $\alpha \Vdash_{\mathcal{K}} \exists_y \varphi(\overline{a}, y) \iff \alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}, b)$ for some element $b \in K_\alpha$
- $\alpha \Vdash_{\mathcal{K}} \forall_y \varphi(\overline{a}, y) \iff \forall_{\alpha \leq^f \alpha'} \alpha' \Vdash_{\mathcal{K}} \varphi(f\overline{a}, b)$ for all $b \in K_{\alpha'}$

Notice that the forcing relation is persistent in the sense that

$$(\alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}) \ \wedge \ \alpha \leq^f \alpha') \ \Rightarrow \ \alpha' \Vdash_{\mathcal{K}} \varphi(f\overline{a})$$

for any formula $\varphi(\overline{x})$. We say that model $\mathcal{K}$ forces the formula $\varphi(\overline{x})$ if it is forced at every its node, i.e.

$$\mathcal{K} \Vdash \varphi \iff \alpha \Vdash_{\mathcal{K}} \varphi \text{ for all } \alpha \in \mathbb{K}.$$

Having given two Kripke models $\mathcal{K}$ and $\mathcal{M}$, the essential question is whether worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$ validate the same formulae. Thus, for nodes $\alpha \in \mathbb{K}$, $\beta \in \mathbb{M}$ we define a relation $\equiv_{p,q,r}$ as follows

$$\alpha \equiv_{p,q,r} \beta \iff (\alpha \Vdash_{\mathcal{K}} \varphi \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi)$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq (^{\to}p, ^{\forall}q, ^{\exists}r)$. We say that α and β are *equivalent*, $\alpha \equiv \beta$, if and only if $\alpha \equiv_{p,q,r} \beta$ for all $p, q, r \geq 0$. For sequences $\overline{a}$ and $\overline{b}$ of elements of worlds K_α and M_β respectively, approaching a more model-theoretical notation, we define

$$(\alpha, \overline{a}) \equiv_{p,q,r} (\beta, \overline{b}) \iff (\alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}) \Leftrightarrow \beta \Vdash_{\mathcal{M}} \varphi(\overline{b}))$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq (^{\to}p, ^{\forall}q, ^{\exists}r)$.

Now, consider any two first-order structures A and B. Let $\overline{a} = a_1, \ldots, a_n$ and $\overline{b} = b_1, \ldots, b_n$ be sequences of elements of A and B respectively. Let $f\colon A \to A'$ and $g\colon B \to B'$ be morphisms of classical structures. To denote the finite mapping $\pi = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ between A and B we will use the symbol $(\overline{a}; \overline{b})$. For morphisms f and g we define a relation $\pi^{f,g} \subseteq A' \times B'$ as $\{(fa_1, gb_1), \ldots, (fa_n, gb_n)\}$. Moreover, we define a *partial isomorphism between structures* A *and* B as a finite mapping $\{(a_1, b_1), \ldots, (a_n, b_n)\} \subseteq A \times B$ such that

$$A \models \varphi(\overline{a}) \iff B \models \varphi(\overline{b})$$

for every atomic formula $\varphi(\overline{x})$.

Finally, we present the definition of bisimulation for first-order Kripke models (for a more general case see [2]). Consider two Kripke models $\mathcal{K}$ and $\mathcal{M}$. Let α and β be nodes of $\mathbb{K}$ and $\mathbb{M}$ respectively, let π range over mappings between worlds of $\mathcal{K}$ and worlds of $\mathcal{M}$, and let $p, q, r \geq 0$. A *bounded bisimulation* between Kripke models $\mathcal{K}$ and $\mathcal{M}$ is a 6-ary relation that satisfies conditions specified below. We will write $\pi\colon \alpha \sim_{p,q,r} \beta$ whenever $\pi, \alpha, p, q, r, \beta$ are in that relation.

(i) $\pi\colon \alpha \sim_{0,0,0} \beta \Longrightarrow \pi$ is a partial isomorphism between K_α and M_β

(ii) $\pi\colon \alpha \sim_{p+1,q,r} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\rightarrow$-zig) for every $\alpha \leq^f \alpha'$ there is $\beta \leq^g \beta'$ such that $\pi^{f,g}\colon \alpha' \sim_{p,q,r} \beta'$

($\rightarrow$-zag) for every $\beta \leq^g \beta'$ there is $\alpha \leq^f \alpha'$ such that $\pi^{f,g}\colon \alpha' \sim_{p,q,r} \beta'$

(iii) $\pi\colon \alpha \sim_{p,q+1,r} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\forall$-zig) for every $\alpha \leq^f \alpha'$ and $a \in K_{\alpha'}$ there are $\beta \leq^g \beta'$ and $b \in M_{\beta'}$ such that $\pi^{f,g} \cup \{(a, b)\}\colon \alpha' \sim_{p,q,r} \beta'$

($\forall$-zag) for every $\beta \leq^g \beta'$ and $b \in M_{\beta'}$ there are $\alpha \leq^f \alpha'$ and $a \in K_{\alpha'}$ such that $\pi^{f,g} \cup \{(a, b)\}\colon \alpha' \sim_{p,q,r} \beta'$

(iv) $\pi\colon \alpha \sim_{p,q,r+1} \beta \Longrightarrow \pi$ is a mapping between K_α and M_β and

($\exists$-zig) for every element $a \in K_\alpha$ there exists $b \in M_\beta$ such that $\pi \cup \{(a, b)\}\colon \alpha \sim_{p,q,r} \beta$

($\exists$-zag) for every element $b \in M_\beta$ there exists $a \in K_\alpha$ such that $\pi \cup \{(a, b)\}\colon \alpha \sim_{p,q,r} \beta$

Bounded Bisimulation and Logical Equivalence

This section reveals the relationship between notions of bounded bisimulation and logical equivalence.

First, we present the well-known result concerning those notions.

THEOREM 1. *Let α and β be nodes of Kripke models $\mathcal{K}$ and $\mathcal{M}$ respectively. Assume $p, q, r \geq 0$ and $(\overline{a}; \overline{b})$ is a mapping between worlds K_α and M_β such that for some bisimulation $\sim$ we have $(\overline{a}; \overline{b})\colon \alpha \sim_{p,q,r} \beta$. Then*

$$\alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}) \iff \beta \Vdash_{\mathcal{M}} \varphi(\overline{b})$$

for every formula $\varphi(\overline{x})$ such that $char(\varphi) \preceq ({}^{\rightarrow}p, {}^{\forall}q, {}^{\exists}r)$.

PROOF: For a proof see [2]. □

A natural question is whether the inverse implication also holds. It turns out that we have to restrict our considerations to a much smaller class of Kripke models. Having analysed corresponding theorems of classical model theory, we have noticed that some additional assumptions on Kripke models, language L or first-order structures are needed.

To start with, the finite signature of L will be considered with no function symbols. Moreover, we introduce the following notions.

DEFINITION 2. We say that model $\mathcal{K}$ is *strongly finite* if and only if both the frame $\mathbb{K}$ and first-order structures assigned to the nodes of $\mathbb{K}$ are finite.

DEFINITION 3. We say that node α of a Kripke model $\mathcal{K}$ is *finitely saturated* with respect to a class of formulas Γ if and only if for every pair of formulas $\varphi(x), \psi(x) \in \Gamma$ whenever there exists a Kripke model $\mathcal{M}$ and a node β of $\mathcal{M}$ such that

$$\alpha \equiv_\Gamma \beta$$

and an element $b \in M_\beta$ with

$$\beta \Vdash_{\mathcal{M}} \varphi(b) \quad \text{and} \quad \beta \nVdash_{\mathcal{M}} \psi(b),$$

then there exists an element $a \in K_\alpha$ with

$$\alpha \Vdash_{\mathcal{K}} \varphi(a) \quad \text{and} \quad \alpha \nVdash_{\mathcal{K}} \psi(a).$$

We will say that Kripke model $\mathcal{K}$ is *finitely saturated* with respect to Γ whenever its nodes are finitely saturated with respect to Γ.

Now we are ready to prove the main result of the paper.

THEOREM 4. *Let $\mathcal{K}$ and $\mathcal{M}$ be strongly finite Kripke models, and let $p, q, r \geq 0$. Moreover, let models $\mathcal{K}$ and $\mathcal{M}$ be finitely saturated with respect to a class of formulas $\Gamma = \{\varphi(\overline{x})\colon char(\varphi) \preceq ({}^{\rightarrow}p + q, {}^{\forall} q, {}^{\exists} r)\}$. Then, for every nodes α and β of models $\mathcal{K}$ and $\mathcal{M}$ respectively, and sequences $\overline{a}$ and $\overline{b}$ of the elements of the structures K_α and M_β respectively, if*

$$(\alpha, \overline{a}) \equiv_{p+q,q,r} (\beta, \overline{b}),$$

then

$$(\overline{a}; \overline{b})\colon \alpha \sim_{p,q,r} \beta$$

for some bisimulation $\sim$.

PROOF: Let α and β be nodes of finite Kripke models $\mathcal{K}$ and $\mathcal{M}$, and let K_α and M_β be finite first-order structures assigned to those nodes respectively. Let $\overline{a}$ and $\overline{b}$ be sequences of the elements of K_α and M_β respectively. We assume that $(\alpha, \overline{a}) \equiv_{p+q,q,r} (\beta, \overline{b})$, i.e.

$$\alpha \Vdash_{\mathcal{K}} \varphi(\overline{a}) \iff \beta \Vdash_{\mathcal{M}} \varphi(\overline{b})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}p+q, {}^{\forall} q, {}^{\exists} r)$. Let $\gamma \geq \alpha$ and $\delta \geq \beta$ be arbitrary nodes accessible from α and β respectively (when it is not necessary and do not lead to confusion, we omit the superscripts f and g that denote the morphisms), and let $\overline{c}$ and $\overline{d}$ be sequences of the elements of K_γ and M_δ respectively. For $0 \leq i \leq p$, $0 \leq j \leq q$, $0 \leq k \leq r$ we define a relation $\sim$ as follows

$$(\overline{c}; \overline{d})\colon \gamma \sim_{i,j,k} \delta \iff (\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \Leftrightarrow \delta \Vdash_{\mathcal{M}} \varphi(\overline{d}))$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}i+j, {}^{\forall} j, {}^{\exists} k)$. By induction on i, j, k we want to prove that $\sim$ is a bounded bisimulation.

(i) First, assume that $(\overline{c}; \overline{d})\colon \gamma \sim_{0,0,0} \delta$, i.e.

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}0, {}^{\forall} 0, {}^{\exists} 0)$. Let $\varphi(\overline{x})$ be an atomic formula. Then, we have

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff K_\gamma \models \varphi(\overline{c}),$$

$$\delta \Vdash_{\mathcal{M}} \varphi(\overline{d}) \iff M_\delta \models \varphi(\overline{d}).$$

So, by the assumption, for all atomic formulae $\varphi(\overline{x})$ we obtain

$$K_\gamma \models \varphi(\overline{c}) \iff M_\delta \models \varphi(\overline{d}).$$

That means that the mapping $(\overline{c};\overline{d})$ is a partial isomorphism between structures K_γ and M_δ.

Now assume that the result holds for some $(i,j,k) \succeq ({}^{\to}0, {}^{\forall}0, {}^{\exists}0)$.

(ii) For $i < p, j \leq q, k \leq r$, assume that $(\overline{c};\overline{d})\colon \gamma \sim_{i+1,j,k} \delta$, i.e.

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\to}i+j+1, {}^{\forall}j, {}^{\exists}k)$. We verify the ($\to$ -zig) property. So, it suffices to show that

$$\forall \gamma \leq^f \gamma' \, \exists \delta \leq^g \delta' \, \big(\gamma' \Vdash_{\mathcal{K}} \varphi(f\overline{c}) \iff \delta' \Vdash_{\mathcal{M}} \varphi(g\overline{d})\big)$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\to}i+j, {}^{\forall}j, {}^{\exists}k)$.

Suppose there exists $\gamma \leq^f \gamma'$ such that for every $\delta \leq^g \delta'$ there exists a formula $\varphi_{\delta'}(\overline{x})$ with $char(\varphi) \preceq ({}^{\to}i+j, {}^{\forall}j, {}^{\exists}k)$ such that

$$(\gamma' \Vdash_{\mathcal{K}} \varphi_{\delta'}(f\overline{c}) \text{ and } \delta' \nVdash_{\mathcal{M}} \varphi_{\delta'}(g\overline{d}))$$

or

$$(\gamma' \nVdash_{\mathcal{K}} \varphi_{\delta'}(f\overline{c}) \text{ and } \delta' \Vdash_{\mathcal{M}} \varphi_{\delta'}(g\overline{d})).$$

Let Θ be the set of all the formulae $\varphi_{\delta'}(\overline{x})$ that have been chosen in the above manner. Note that, since $\delta' \geq \delta$ range over the finite Kripke model $\mathcal{M}$, the set Θ is finite too. Consider the two following subsets of Θ

$$\Theta_0 = \{\varphi_{\delta'}(\overline{x}) \in \Theta \colon \gamma' \Vdash_{\mathcal{K}} \varphi_{\delta'}(f\overline{c}) \text{ and } \delta' \nVdash_{\mathcal{M}} \varphi_{\delta'}(g\overline{d})\}$$

and

$$\Theta_1 = \{\varphi_{\delta'}(\overline{x}) \in \Theta \colon \gamma' \nVdash_{\mathcal{K}} \varphi_{\delta'}(f\overline{c}) \text{ and } \delta' \Vdash_{\mathcal{M}} \varphi_{\delta'}(g\overline{d})\}.$$

Obviously, the sets Θ_0 and Θ_1 cannot be both empty. Let us define a formula

$$\theta = \begin{cases} \bigwedge \Theta_0 & \text{if} \quad \Theta_1 = \emptyset \\ \bigwedge \Theta_0 \to \bigvee \Theta_1 & \text{if} \quad \Theta_0 \neq \emptyset, \Theta_1 \neq \emptyset \\ \bigvee \Theta_1 & \text{if} \quad \Theta_0 = \emptyset \end{cases}$$

Let us note that when $\Theta_1 = \emptyset$, it can be easily seen that $\gamma' \Vdash_{\mathcal{K}} \theta(f\overline{c})$, and hence, $\gamma \nVdash_{\mathcal{K}} \neg\theta(\overline{c})$. Moreover, $\delta' \nVdash_{\mathcal{M}} \theta(g\overline{d})$ for all $\delta \leq^g \delta'$, so $\delta \Vdash_{\mathcal{M}} \neg\theta(\overline{d})$, which is contrary to our assumption, since $char(\neg\theta) \preceq ({}^{\to}i+j+1, {}^{\forall}j, {}^{\exists}k)$.

Similarly, when $\Theta_0 = \emptyset$, we have $\gamma' \nVdash_{\mathcal{K}} \theta(f\overline{c})$, thus $\gamma \nVdash_{\mathcal{K}} \theta(\overline{c})$. Moreover, $\delta' \Vdash_{\mathcal{M}} \theta(g\overline{d})$ for all $\delta \leq^g \delta'$, so, in particular, $\delta \Vdash_{\mathcal{M}} \theta(\overline{d})$. Then, since $char(\theta) \preceq ({}^{\rightarrow}i + j, {}^{\forall}j, {}^{\exists}k)$, we get immediately a contradiction to our assumption.

So, let us consider the case when $\Theta_0 \neq \emptyset$ and $\Theta_1 \neq \emptyset$. Notice that

$$\gamma' \Vdash_{\mathcal{K}} \bigwedge \Theta_0(f\overline{c}) \quad \text{and} \quad \gamma' \nVdash_{\mathcal{K}} \bigvee \Theta_1(f\overline{c}) \tag{1}$$

and

$$\delta' \nVdash_{\mathcal{M}} \bigwedge \Theta_0(g\overline{d}) \quad \text{and} \quad \delta' \Vdash_{\mathcal{M}} \bigvee \Theta_1(g\overline{d}) \tag{2}$$

for every $\delta \leq^g \delta'$. Consider the formula $(\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(\overline{x})$. Note that $char\big(\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1\big) \preceq ({}^{\rightarrow}i + j + 1, {}^{\forall}j, {}^{\exists}k)$. By (1) we get

$$\gamma \nVdash_{\mathcal{K}} (\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(\overline{c})$$

and, by (2), we get

$$\delta \Vdash_{\mathcal{M}} (\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(\overline{d}),$$

which is contrary to the assumption

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}i + j + 1, {}^{\forall}j, {}^{\exists}k)$.

(iii) Now, for $i \leq p, j < q, k \leq r$ assume that $(\overline{c}; \overline{d})\colon \gamma \sim_{i,j+1,k} \delta$, i.e

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}i + j + 1, {}^{\forall}j + 1, {}^{\exists}k)$. We verify the ($\forall$-zig) property. It suffices to show that for every $\gamma \leq^f \gamma'$ and every element $c \in K_{\gamma'}$ there exists $\delta \leq^g \delta'$ and $d \in M_{\delta'}$ such that

$$\gamma' \Vdash_{\mathcal{K}} \varphi(f\overline{c}, c) \iff \delta' \Vdash_{\mathcal{M}} \varphi(g\overline{d}, d)$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\rightarrow}i + j, {}^{\forall}j, {}^{\exists}k)$. For simplicity we will suppress the parameters $f\overline{c}$ and $g\overline{d}$.

Let us suppose there exists $\gamma \leq^f \gamma'$ and an element $c \in K_{\gamma'}$ such that for every $\delta \leq^g \delta'$ and every element $d \in M_{\delta'}$ there exists a formula $\varphi_{\delta',d}(x)$ with $char(\varphi) \preceq ({}^{\rightarrow}i + j, {}^{\forall}j, {}^{\exists}k)$ such that

$$(\gamma' \Vdash_{\mathcal{K}} \varphi_{\delta',d}(c) \text{ and } \delta' \nVdash_{\mathcal{M}} \varphi_{\delta',d}(d))$$

or

$$(\gamma' \nVdash_{\mathcal{K}} \varphi_{\delta',d}(c) \text{ and } \delta' \Vdash_{\mathcal{M}} \varphi_{\delta',d}(d)).$$

As previously, let Θ denote the set of all formulae $\varphi_{\delta',d}(x)$ which have been chosen in that manner. Since $\delta' \geq \delta$ and d range over the finite Kripke model $\mathcal{M}$ and the finite structure $M_{\delta'}$ respectively, the set Θ is finite too. So, consider the two following subsets of Θ

$$\Theta_0 = \{\varphi_{\delta',d}(x) \in \Theta \colon \gamma' \Vdash_{\mathcal{K}} \varphi_{\delta',d}(c) \text{ and } \delta' \nVdash_{\mathcal{M}} \varphi_{\delta',d}(d)\}$$

and

$$\Theta_1 = \{\varphi_{\delta',d}(x) \in \Theta \colon \gamma' \nVdash_{\mathcal{K}} \varphi_{\delta',d}(c) \text{ and } \delta' \Vdash_{\mathcal{M}} \varphi_{\delta',d}(d)\}.$$

We define formula θ as previously. First, let us notice that when $\Theta_1 = \emptyset$, then $\gamma' \Vdash_{\mathcal{K}} \bigwedge \Theta_0(c)$. So, it is easy to see that $\gamma' \nVdash_{\mathcal{K}} \neg \bigwedge \Theta_0(c)$, and, since $\gamma' \geq \gamma$ and the element $c \in K_{\gamma'}$ were fixed, we get $\gamma \nVdash_{\mathcal{K}} \forall_x \neg\theta(x)$. Moreover, $\delta' \nVdash_{\mathcal{M}} \bigwedge \Theta_0(d)$ for every $\delta' \geq \delta$ and every $d \in M_{\delta'}$, which means that $\delta' \Vdash_{\mathcal{M}} \neg \bigwedge \Theta_0(d)$ for every $\delta' \geq \delta$ and every element $d \in M_{\delta'}$. And hence, $\delta \Vdash_{\mathcal{M}} \forall_x \neg\theta(x)$, which is contrary to our assumption, since $char(\forall_x \neg\theta) \preceq ({}^{\rightarrow}i + j + 1, {}^{\forall}j + 1, {}^{\exists}k)$.

Similarly, when $\Theta_0 = \emptyset$, we get $\gamma' \nVdash_{\mathcal{K}} \bigvee \Theta_1(c)$, and thus $\gamma \nVdash_{\mathcal{K}} \forall_x \theta(x)$. Moreover, $\delta' \Vdash_{\mathcal{M}} \bigvee \Theta_1(d)$ for every $\delta' \geq \delta$ and every $d \in M_{\delta'}$. Then, $\delta \Vdash_{\mathcal{M}} \forall_x \theta(x)$. But since $char(\forall_x \theta) \preceq ({}^{\rightarrow}i + j, {}^{\forall}j + 1, {}^{\exists}k)$, we get a contradiction to our assumption.

Now, let us consider the case when $\Theta_0 \neq \emptyset$ and $\Theta_1 \neq \emptyset$. Since

$$\gamma' \Vdash_{\mathcal{K}} \bigwedge \Theta_0(c) \quad \text{and} \quad \gamma' \nVdash_{\mathcal{K}} \bigvee \Theta_1(c),$$

we get

$$\gamma \nVdash_{\mathcal{K}} \forall_y (\bigwedge \Theta_0 \rightarrow \bigvee \Theta_1)(y).$$

Moreover, since for every $\delta' \geq \delta$ and every $d \in M_{\delta'}$

$$\delta' \nVdash_{\mathcal{M}} \bigwedge \Theta_0(d) \quad \text{and} \quad \delta' \Vdash_{\mathcal{M}} \bigvee \Theta_1(d),$$

we obtain

$$\delta \Vdash_{\mathcal{M}} \forall_y(\bigwedge \Theta_0 \to \bigvee \Theta_1)(y).$$

But since $char\big(\forall_y(\bigwedge \Theta_0 \to \bigvee \Theta_1)\big) \preceq ({}^{\to}i + j + 1, {}^{\forall} j + 1, {}^{\exists} k)$, we get a contradiction to the assumption $(\overline{c}; \overline{d})\colon \gamma \sim_{i,j+1,k} \delta$.

(iv) To finish the proof, for $i \leq p, j \leq q, k < r$ assume that $(\overline{c}; \overline{d})\colon \gamma \sim_{i,j,k+1} \delta$, i.e

$$\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d})$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\to}i + j, {}^{\forall} j, {}^{\exists} k + 1)$. We verify the ($\exists$-zig) property. So, we have to show that

$$\forall c \in K_\gamma\, \exists d \in M_\delta\, \big(\gamma \Vdash_{\mathcal{K}} \varphi(\overline{c}, c) \iff \delta \Vdash_{\mathcal{M}} \varphi(\overline{d}, d)\big)$$

for all formulae $\varphi(\overline{x})$ with $char(\varphi) \preceq ({}^{\to}i + j, {}^{\forall} j, {}^{\exists} k)$. Once again, for simplicity, we will suppress the parameters $\overline{c}$ and $\overline{d}$.

Let $c \in K_\gamma$. Suppose such an element $d \in M_\delta$ does not exist. Then, for every $d \in M_\delta$ there exists a formula $\varphi_d(x)$ with $char(\varphi) \preceq ({}^{\to}i + j, {}^{\forall} j, {}^{\exists} k)$ such that

$$(\gamma \Vdash_{\mathcal{K}} \varphi_d(c) \text{ and } \delta \nVdash_{\mathcal{M}} \varphi_d(d))$$

or

$$(\gamma \nVdash_{\mathcal{K}} \varphi_d(c) \text{ and } \delta \Vdash_{\mathcal{M}} \varphi_d(d)).$$

Let Θ be the set of all the formulae $\varphi_d(x)$. Again, since the structure M_δ is finite, the set Θ is finite too. So, consider the two following subsets of Θ

$$\Theta_0 = \{\varphi_d(x) \in \Theta\colon \gamma \Vdash_{\mathcal{K}} \varphi_d(c) \text{ and } \delta \nVdash_{\mathcal{M}} \varphi_d(d)\}$$

and

$$\Theta_1 = \{\varphi_d(x) \in \Theta\colon \gamma \nVdash_{\mathcal{K}} \varphi_d(c) \text{ and } \delta \Vdash_{\mathcal{M}} \varphi_d(d)\}.$$

Moreover, let us define the following subsets of M_δ

$$T_0 = \{d \in M_\delta\colon \varphi_d \in \Theta_0\},$$
$$T_1 = \{d \in M_\delta\colon \varphi_d \in \Theta_1\}.$$

First, notice that

$$d \in T_0 \iff \delta \nVdash_{\mathcal{M}} \bigwedge \Theta_0(d) \tag{3}$$

and

$$d \in T_1 \iff \delta \Vdash_{\mathcal{M}} \bigvee \Theta_1(d) \tag{4}$$

for every $d \in M_\delta$. Note also that according to our assumption $T_0 \cup T_1 = M_\delta$.

Let us consider the case when $\Theta_1 = \emptyset$. That means $M_\delta = T_0$. We have $\gamma \Vdash_{\mathcal{K}} \bigwedge \Theta_0(c)$, and hence $\gamma \Vdash_{\mathcal{K}} \exists_x \bigwedge \Theta_0(x)$. Since $\gamma \equiv_{i+j,j,k+1} \delta$, then $\delta \Vdash_{\mathcal{M}} \exists_x \bigwedge \Theta_0(x)$. But, by (3), $\delta \nVdash_{\mathcal{M}} \bigwedge \Theta_0(d)$ for each $d \in T_0 = M_\delta$, and hence $\delta \nVdash_{\mathcal{M}} \exists_x \bigwedge \Theta_0(x)$, a contradiction.

Now assume that $\Theta_0 = \emptyset$. Then $M_\delta = T_1$. Let $\psi(x) \in \Gamma$ be any formula provable in intuitionistic predicate logic. Then

$$\gamma \Vdash_{\mathcal{K}} \psi(c) \quad \text{and} \quad \gamma \nVdash_{\mathcal{K}} \bigvee \Theta_1(c).$$

By the assumption $\gamma \equiv_{i+j,j,k+1} \delta$. Because node δ is finitely saturated with respect to Γ, there exists an element $d \in M_\delta$ such that

$$\delta \Vdash_{\mathcal{M}} \psi(d) \quad \text{and} \quad \delta \nVdash_{\mathcal{M}} \bigvee \Theta_1(d).$$

Hence, by (4), $d \notin T_1 = M_\delta$, a contradiction.

So, we can consider the case when $\Theta_0 \neq \emptyset$ and $\Theta_1 \neq \emptyset$. We have,

$$\gamma \Vdash_{\mathcal{K}} \bigwedge \Theta_0(c) \quad \text{and} \quad \gamma \nVdash_{\mathcal{K}} \bigvee \Theta_1(c).$$

By the assumption $\gamma \equiv_{i+j,j,k+1} \delta$. Notice also that $char(\bigwedge \Theta_0), char(\bigvee \Theta_1) \preceq (^{\rightarrow} i+j, ^{\forall} j, ^{\exists} k)$, so $\bigwedge \Theta_0, \bigvee \Theta_1 \in \Gamma$. Then, since δ is finitely saturated with respect to Γ, there exists an element $d \in M_\delta$ with

$$\delta \Vdash_{\mathcal{M}} \bigwedge \Theta_0(d) \quad \text{and} \quad \delta \nVdash_{\mathcal{M}} \bigvee \Theta_1(d).$$

Hence, by (3) and (4), we obtain

$$d \notin T_0 \quad \text{and} \quad d \notin T_1.$$

And thus, $T_0 \cup T_1 \neq M_\delta$, which contradicts our assumption. □

Combining Theorem 1 and Theorem 4 we get the following fact.

THEOREM 5. *Let $\mathcal{K}$ and $\mathcal{M}$ be the strongly finite Kripke models, and let $p, q, r \geq 0$. Moreover, let models $\mathcal{K}$ and $\mathcal{M}$ be finitely saturated with respect to the set Γ. Consider nodes α and β of models $\mathcal{K}$ and $\mathcal{M}$ respectively, and sequences $\overline{a}$ and $\overline{b}$ of the elements of the structures K_α and M_β respectively. Then,*

$$(\alpha, \overline{a}) \equiv_{p+q,q,r} (\beta, \overline{b}),$$

if and only if

$$(\overline{a}; \overline{b})\colon \alpha \sim_{p,q,r} \beta \text{ for some bisimulation } \sim .$$

Hence, as a "limit case" we obtain the following corollary.

COROLLARY 6. *Let $\mathcal{K}$ and $\mathcal{M}$ be finitely saturated, strongly finite Kripke models, and let α and β be nodes of models $\mathcal{K}$ and $\mathcal{M}$ respectively. Then,*

$$\alpha \equiv \beta \textit{ if and only if } \alpha \sim \beta.$$

As we could notice in the proof of Theorem 4, the assumption that the models $\mathcal{K}$ and $\mathcal{M}$ are finitely saturated with respect to Γ is used only in the case of the ($\exists$-zig) property. In particular, the theorem remains true without that assumption when we restrict ourselves to the language L without the existential quantifier. More precisely, we have the following fact.

THEOREM 7. *Consider finite language L with no function symbols, and without the existential quantifier. Let $\mathcal{K}$ and $\mathcal{M}$ be the strongly finite Kripke models for the language L, and let $p, q \geq 0$. Then, for nodes α and β of models $\mathcal{K}$ and $\mathcal{M}$ respectively, and sequences $\overline{a}$ and $\overline{b}$ of the elements of the structures K_α and M_β respectively, if*

$$(\alpha, \overline{a}) \equiv_{(\to p+q, \forall q)} (\beta, \overline{b}),$$

then

$$(\overline{a}; \overline{b})\colon \alpha \sim_{(\to p, \forall q)} \beta \textit{ for some bisimulation } \sim .$$

PROOF: Follows from the proof of Theorem 4. □

COROLLARY 8. *Consider finite language L with no function symbols, and without the existential quantifier. Let $\mathcal{K}$ and $\mathcal{M}$ be strongly finite Kripke models for the language L, and let α and β be nodes of models $\mathcal{K}$ and $\mathcal{M}$ respectively. Then,*

$$\alpha \equiv \beta \textit{ if and only if } \alpha \sim \beta.$$

REMARK 9. In the proof of Theorem 4, instead of finite saturation we could also assume that the Kripke models in question satisfy the Law of Excluded Middle, with respect to the class Γ.

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